NEGATIVE GENERATORS OF THE VIRASORO CONSTRAINTS FOR THE BKP HIERARCHY

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Abstract. We give a straightforward derivation of the string equation and Virasoro constraints on the τ function of the BKP hierarchy by means of some special additional symmetry flows. The explicit forms of the actions of these additional symmetry flows on the wave function and then the negative Virasoro generators L_{-k} are given, where k is a positive integer.

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1. Introduction

Since its introduction in a very convenient form in 1986 [1] and stimulated by the importance of the string equation [2], much attention has been paid to the study of the additional symmetries and the Virasoro constraints [3–5] for the Kadomtsev-Petviashvili(KP) hierarchy [6,7]. For this hierarchy, it is now well established that there are two representations of the additional symmetries, i.e. Sato vertex operator form [6] and Orlov -Schulmann(OS) M-operator form [1]. In 1994, they were proved to be equivalent by the action on the wave functions of the KP hierarchy in the two different forms [8,9]. In this process, Adler-Shiota-van Moerbeke(ASvM) formula plays a crucial role. Almost at the same time Dickey presented a very elegant and compact proof of ASvM formula [10] based on the Lax operator L and OS's M operator. In addition he also [11] derived the string equation, the action of the additional symmetries on the τ function and the Virasoro constraints of the KP hierarchy.

The BKP hierarchy [6,12] is a reductional sub-hierarchy of the KP with a restriction on the Lax operator $L^* = -\partial L \partial^{-1}$ (here $*$ stands for a formal adjoint operation, L is a Lax operator of the BKP hierarchy). Therefore, it was natural to expect intensive investigations on the additional symmetry and its associated structures of the BKP hierarchy after the discovery of this kind of symmetries on the KP hierarchy. In this context, Johan [13,14] has obtained the Virasoro constraints on the τ function and ASvM formula of the BKP hierarchy by an algebraic method. Takasaki [15] found appropriate restrictions on the generators of the additional symmetries for the BKP hierarchy. Very recently, by using Takasaki's result [15] and Dickey's method [11], Tu [16] has given an explicit form of the generators of the additional symmetries, and then an alternative proof of the ASvM formula for the BKP hierarchy. This new proof is more simpler and transparent in comparison with the algebraic method presented in [14]. Here it is important to mention that due to the reductional conditions of the BKP hierarchy many differences in relation to the KP hierarchy may emerge, turning this investigation highly non-trivial. For example, the generators of the additional symmetries of the BKP hierarchy are

also correspondingly restricted [15](specifically,see eq.(40)). This fact implies that the generators of the additional symmetries for the BKP hierarchy must be different compared to their counterparts on the KP hierarchy. In this scenario, it would be relevant to derive the Virasoro constraints of the BKP hierarchy using also the potentialities of the Dickey's method.

In this work, applications of the additional symmetries of the BKP hierarchy are studied in detail. In particular, we find the string equation and negative generators of Virasoro constraints on the τ function for the BKP hierarchy by means of the additional symmetry flows. We also give the explicit forms of the negative Virasoro generators by calculating the action of the additional symmetry flows on the τ function, which is induced by the action of the additional symmetry flows on the wave function of the BKP hierarchy.

The organization of this paper is as follows. In section 2 we present a brief summary of the BKP and its additional symmetry, which is followed by string equation and some special additional symmetry flow equations in section 3. In section 4 we derive the negative generators of the Virasoro constraints. Section 5 is devoted to conclusions and discussions.

2. BKP Hierarchy and its additional symmetries

Let L be the pseudo-differential operator,

$$L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + u_3\partial^{-3} + \cdots , \tag{2.1}$$

and then the KP hierarchy is defined by the set of partial differential equations u_i with respect to independent variables t_j

$$\frac{\partial L}{\partial t_n} = [B_n, L],\ n = 1, 2, 3, \cdots . \tag{2.2}$$

Here $B_n = (L^n)_+ = \sum_{k=0}^{n} a_k\partial^k$ denotes the non-negative powers of ∂ in L^n, $\partial = \partial/\partial_x$, $u_i = u_i(x = t_1, t_2, t_3, \cdots ,)$. The other notation $L^n_- = L^n - L^n_+$ will be needed by the sequent text. L is called the Lax operator and eq.(2.2) is called the Lax equation of the KP hierarchy. In order to define the BKP hierarchy, we need a formal adjoint operation $*$ for an arbitrary pseudo-differential operator $P = \sum_i p_i\partial^i$, $P^* = \sum_i(-1)^i\partial^i p_i$. For example, $\partial^* = -\partial$, $(\partial^{-1})^* = -\partial^{-1}$, and $(AB)^* = B^*A^*$ for two operators. The BKP hierarchy [6, 12] is a reduction of the KP hierarchy by the constraint

$$L^* = -\partial L\partial^{-1}, \tag{2.3}$$

which compresses all even flows of the KP hierarchy, i.e. the Lax equation of the BKP hierarchy has only odd flows ,

$$\frac{\partial L}{\partial t_{2n+1}} = [B_{2n+1}, L],\ n = 0, 1, 2, \cdots . \tag{2.4}$$

Thus $u_i = u_i(t_1, t_3, t_5, \cdots)$ for the BKP hierarchy.

The Lax equation of the BKP hierarchy can be given by the consistent conditions of the following set of linear partial differential equations

$$Lw(t, \lambda) = \lambda w(t, \lambda), \frac{\partial w(t, \lambda)}{\partial t_{2n+1}} = B_{2n+1}w(t, \lambda),\ t = (t_1, t_3, t_5, \cdots). \tag{2.5}$$

Here $w(t, \lambda)$ is identified as a wave function. Let ϕ be the wave operator(or Sato operator) of the BKP hierarchy $\phi = 1 + \sum_{i=1}^{\infty} w_i\partial^{-i}$, then the Lax operator and the wave function admit the following representation

$$L = \phi\partial\phi^{-1},\quad w(t, \lambda) = \phi(t)e^{\xi(t,\lambda)} = \hat{w}e^{\xi(t,\lambda)}, \tag{2.6}$$

in which $\xi(t,\lambda)=\lambda t_1+\lambda^3 t_3+\cdots+\lambda^{2n+1}t_{2n+1}+\cdots$, $\hat{w}=1+\frac{w_1}{\lambda}+\frac{w_2}{\lambda^2}+\frac{w_3}{\lambda^3}+\cdots$. Similar to the KP hierarchy, the BKP hierarchy also has a sole function,τ function $\tau(t)=\tau(t_1,t_3,t_5,\cdots,t_{2n-1}\cdots)$ (n is a positive integer), such that all of the dynamical coordinates u_i can be expressed, and further the wave function is

$$w(t,\lambda)=\hat{w}(t,\lambda)e^{\xi(t,\lambda)}=\frac{\tau(t_1-\frac{2}{\lambda},t_3-\frac{2}{3\lambda^3},t_5-\frac{2}{5\lambda^5},\cdots)}{\tau(t)}e^{\xi(t,\lambda)}\equiv\frac{\tilde{\tau}(t,\lambda)}{\tau(t)}e^{\xi(t,\lambda)} \tag{2.7}$$

It is easy to show that the Lax equation is equivalent to Sato equation

$$\frac{\partial\phi}{\partial t_{2n+1}}=-L_-^{2n+1}\phi, \tag{2.8}$$

and the constraint on L in eq.(2.3) is transformed to the constraint on the wave operator

$$\phi^*=\partial\phi^{-1}\partial^{-1}. \tag{2.9}$$

Eq.(2.9) is a crucial condition to construct the additional symmetries of the BKP hierarchy, which will affect the action of the additional symmetry on the operator ϕ. It leads to a distinct explicit form of the generators of the additional symmetry in comparison to the cases of the KP hierarchy [1,7] and the CKP hierarchy [17], as we shall see latter.

Now we recall the additional symmetries given by Tu [16] of the BKP hierarchy. Let the OS's operator M be given by

$$M=\phi\Gamma\phi^{-1},\ \Gamma=\sum_{i=1}^{\infty}(2i-1)t_{2i-1}\partial^{2i-2}=t_1+3t_3\partial^2+5t_5\partial^4+\cdots, \tag{2.10}$$

then they satisfy the useful technical identities

$$[M,L^l]=-lL^{l-1},l\in Z, \tag{2.11}$$

$$[M^m,L]=-mM^{m-1},m\in Z_+. \tag{2.12}$$

Define the additional flows

$$\frac{\partial\phi}{\partial t^*_{m,l}}=-\left(A_{m,l}\right)_-\phi, \tag{2.13}$$

or equivalently

$$\frac{\partial L}{\partial t^*_{m,l}}=-[(A_{m,l})_-,L], \tag{2.14}$$

where $A_{m,l}=A_{m,l}(L,M)$ are monomials in L and M. As pointed in the last paragraph, constraints on L in eq.(2.3), or equivalently on ϕ in eq.(2.9) imply restrictions on the generators, and then one distinct form of $A_{m,l}$ [16] is

$$A_{m,l}=M^mL^l-(-1)^lL^{l-1}M^mL. \tag{2.15}$$

Indeed, this generator is different compared to results $A_{m,l}=M^mL^l$ [1,7] for the KP hierarchy and $A_{m,l}=M^mL^l-(-1)^lL^lM^m$ [17] for the CKP hierarchy.

Proposition 1. *([16])1) The additional flows are symmetries of the BKP hierarchy. 2)They form a centerless $W^B_{1+\infty}$-algebra understanding their actions on ϕ as eq.(2.13).*

3. Some special additional symmetry flow equations

We further concentrate on some special additional symmetry flows in order to find suitable additional flows implying the Virasoro constraints on the τ function of the BKP hierarchy. So two examples are calculated in the following.

Proposition 2. *The action on L of the additional flows associated with $A_{1,l} = -(l-1)L^{l-1}$ is in the form of*

$$\partial_{t^*_{1,l}} L = (l-1)[(L^{l-1})_-, L] = \begin{cases} 0, \ \ for\ l = 0, -2, -4, -6, \cdots . \\ -(l-1)(\partial_{t_{1-l}} L), \ \ for\ l = 2, 4, 6, \cdots . \end{cases} \tag{3.1}$$

Although this result is different with its counterpart in the KP hierarchy, this case is not interesting enough because this additional symmetry flows are almost equivalent to the CKP flows acing on the space of the Lax operators L. The reason is that l is an even integer.

Therefore we consider $A_{1,-(l-1)}$, and calculate its action on L^l. For this end, from now on assume that $l = 2k$ and k is a positive integer. By using eq.(2.11), the $A_{1,-(l-1)}$ can be expressed as

$$A_{1,-(l-1)} = 2ML^{-(l-1)} - lL^{-l}, \tag{3.2}$$

and then

$$\begin{aligned} \partial_{t^*_{1,-(l-1)}} L^l &= -[(A_{1,-(l-1)})_-, L^l] \\ &= [(A_{1,-(l-1)})_+, L^l] + [-(A_{1,-(l-1)}), L^l] = [(A_{1,-(l-1)})_+, L^l] + 2l. \end{aligned} \tag{3.3}$$

Thus we get the following proposition based on the actions of the additional symmetry $A_{1,-(l-1)}$ on the L^l.

Proposition 3. *Let $l = 2m(2n+1)$, $m, n = 1, 2, 3, \cdots$, and L^l is independent of $t^*_{1,-(l-1)}$, then the string equation of the BKP hierarchy is*

$$[L^l, \frac{1}{2l}(A_{1,-(l-1)})_+] = 1. \tag{3.4}$$

Furthermore, this equation can be written in a more explicit form as follows,

$$[L^{2k}, \frac{1}{2k} ML^{-(2k-1)} - \frac{1}{2} L^{-2k}] = 1, \ \ k = m(2n+1). \tag{3.5}$$

Proof The eq.(3.3) and $\partial_{t^*_{1,-(l-1)}} L^l = 0$ deduce directly eq.(3.4). Moreover $\partial_{t^*_{1,-(l-1)}} L^l = 0$ infers $(A_{1,-(l-1)})_- = 0$, and then $(ML^{-(2k-1)})_- = kL^{-2k}$ and $(A_{1,-(2k-1)})_+ = 2ML^{-(2k-1)} - 2kL^{-2k}$. Taking this back into eq.(3.5), then eq.(3.5) is proved, which completes the proof. □

Note that eq.(3.5) was also obtained by Johan [13] from the Virasoro constraints on the τ function of the BKP hierarchy. However, his equation is not the string equation without the restrictions of l. In other words, L^l can not equal $(L^l)_+$ with an arbitrary positive even integer l.

Corollary 1. *If L^l satisfy the eq.(3.4),then*

$$-\frac{1}{2k} \sum_{n \geq k+1} (2n-1) t_{2n-1} (\partial_{t_{2n-(2k+1)}} L^{2k}) = 1. \tag{3.6}$$

Let k = 1, the zero order terms of above equation tell us

$$\frac{1}{2} \sum_{n \geq 2} (2n-1) t_{2n-1} (\partial_{t_{2n-3}} \tau) + \frac{1}{8} x^2 \tau = 0. \tag{3.7}$$

This result is indeed distinct with the case of KP hierarchy given by corollary 1.2 of Ref. [18].
Proof By a direct calculation, the left hand side of eq.(3.4) becomes

$$1=\left[L^{2k},\frac{1}{2k}(ML^{-(2k-1)})_{+}\right]=\left[L^{2k},\frac{1}{2k}\left(\phi\sum_{n=1}^{\infty}(2n-1)t_{2n-1}\partial^{2n-2k-1}\phi^{-1}\right)_{+}\right]$$
$$=\left[L^{2k},\frac{1}{2k}\left(\phi\sum_{n\geq k+1}^{\infty}(2n-1)t_{2n-1}\partial^{2n-2k-1}\phi^{-1}\right)_{+}\right].$$

Note that the change in index of summation is due to the identity $(\phi\partial^{-m}\phi^{-1})_{+}=0$, here m is a positive integer. We also should note $L_{+}^{k}=(\phi\partial^{k}\phi^{-1})_{+}$ with $k\geq 0$, and then get

$$1=\left[L^{2k},\frac{1}{2k}\sum_{n\geq k+1}^{\infty}(2n-1)t_{2n-1}(L^{2n-2k-1})_{+}\right]=-\frac{1}{2k}\sum_{n\geq k+1}^{\infty}(2n-1)t_{2n-1}(\partial_{t_{2n-2k-1}}L^{2k}),$$

which is eq.(3.6). Furthermore, let $k=1$, taking $L^{2k}=L^{2}=\partial^{2}+2u_{1}+$lower order terms and $u_{1}=2(\ln\tau)_{xx}$ back into eq. (3.6), we get the zero order terms in both sides,

$$-\frac{1}{2}\sum_{n\geq 2}(2n-1)t_{2n-1}(4\partial_{t_{2n-3}}(\ln\tau)_{xx})=1.$$

By exchanging the order of the derivative with respect to x and t_{2n-3}, then

$$-2\sum_{n\geq 2}(2n-1)t_{2n-1}\left(\frac{1}{\tau}(\partial_{t_{2n-3}}\tau)\right)_{xx}=1.$$

Integrating the above formula two times on x and choosing suitable constants, then eq.(3.7) is reached, and thus completes the proof. □

4. Virasoro generators

It is easy to find that eq.(3.4) is equivalent to $\partial_{t^{*}_{1,-(l-1)}}\phi=0$, with $l=2k$. To get the Virasoro constraints on the τ function and the Virasoro generators, firstly we shall pass the action of the flows $\partial_{t^{*}_{1,-(l-1)}}$ on the wave operator ϕ to the action on the wave function w, and then on the τ function of the BKP hierarchy. In this context, $\hat{w}(t,z)$ plays the role of a bridge connecting actions on the wave operator ϕ and on the τ function. The following lemmas are necessary to do this.

Lemma 1. *For $l=2k,\ k=1,2,3,4,\cdots,$*

$$(A_{1,-(l-1)})_{-}=2\phi\left(\sum_{n=1}^{k}(2n-1)t_{2n-1}\partial^{2(n-k)-1}\right)\phi^{-1}$$
$$+2\sum_{n=k+1}^{\infty}(2n-1)t_{2n-1}L_{-}^{2(n-k)-1}-lL^{-l} \tag{4.1}$$

Proof According to the definitions of M and L,

$$(ML^{-(l-1)})_{-}=(\phi\Gamma\phi^{-1}\phi\partial^{-(l-1)}\phi^{-1})_{-}=(\phi\Gamma\partial^{-(l-1)}\phi^{-1})_{-}$$
$$=\left(\phi(\sum_{n=1}^{k}(2n-1)t_{2n-1}\partial^{2n-2k-1}+\sum_{n=k+1}^{\infty}(2n-1)t_{2n-1}\partial^{2n-2k-1})\phi^{-1}\right)_{-}$$

$$= \phi\left(\sum_{n=1}^{k}(2n-1)t_{2n-1}\partial^{2n-2k-1}\right)\phi^{-1} + \left(\sum_{n=k+1}^{\infty}(2n-1)t_{2n-1}\phi\partial^{2n-2k-1}\phi^{-1}\right)_{-}$$

$$= \phi\left(\sum_{n=1}^{k}(2n-1)t_{2n-1}\partial^{2n-2k-1}\right)\phi^{-1} + \sum_{n=k+1}^{\infty}(2n-1)t_{2n-1}L_{-}^{2(n-k)-1}$$

In the second term of the last second equality, ϕ pass the t_{2n-1} because ϕ is involved only with ∂_{t_1}, but there $2n-1>1$. Thus, taking this representation of $(ML^{-(l-1)})_{-}$ into the generator $A_{1,-(l-1)} = 2ML^{-(l-1)} - lL^{-1}$, and then the lemma is proved. □

Proposition 4. *Let $l = 2k$ as before, and $\hat{w}(t,z)$ is given by eq.(2.7), then*

$$\partial_{t^*_{1,-(l-1)}}\hat{w}(t,z) = -2\left(z^{-2k+1}(\frac{\partial}{\partial z}\hat{w}) + \sum_{n=1}^{k}(2n-1)t_{2n-1}z^{2n-2k-1}\hat{w}\right) + 2\sum_{n=k+1}^{\infty}(2n-1)t_{2n-1}\frac{\partial\hat{w}}{\partial t_{2n-2k-1}} + 2kz^{-2k}\hat{w}. \tag{4.2}$$

Proof First of all, by using lemma 1, the additional symmetry flow acts on the wave operator ϕ as

$$\partial_{t^*_{1,-(l-1)}}\phi = -(A_{1,-(l-1)})_{-}\phi = -2\phi\sum_{n=1}^{k}(2n-1)t_{2n-1}\partial^{2(n-k)-1} +2\sum_{n=k+1}^{\infty}(2n-1)t_{2n-1}(\partial_{t_{2(n-k)-1}}\phi) + l\phi\partial^{-l}.$$

Note that this is an operator equation, thus we can apply the function e^{xz} to both side simultaneously. Therefore, by applying to both sides of the last formula e^{xz} and using two identities: $[\phi,x]e^{xz} = (\frac{\partial}{\partial z}\hat{w})e^{xz}$ and $\phi\partial^{-l}e^{xz} = \phi z^{-l}e^{xz} = z^{-l}\hat{w}e^{xz}$, we achieve that

$$\begin{aligned}(\partial_{t^*_{1,-(l-1)}}\hat{w})e^{xz} &= -((A_{1,-(l-1)})_{-}\phi)e^{xz}\\ &= -2\phi(xz^{-2k+1}e^{xz} + \sum_{n=2}^{k}(2n-1)t_{2n-1}z^{2n-2k-1}e^{xz})\\ &\quad +2\sum_{n=k+1}^{\infty}(2n-1)t_{2n-1}(\partial_{t_{2(n-k)-1}}\hat{w})e^{xz} + 2kz^{-2k}\hat{w}e^{xz}\\ &= -2(z^{-2k+1}(\frac{\partial}{\partial z}\hat{w}) + \sum_{n=2}^{k}(2n-1)t_{2n-1}z^{2n-2k-1}\hat{w})e^{xz}\\ &\quad +2\sum_{n=k+1}^{\infty}(2n-1)t_{2n-1}(\partial_{t_{2(n-k)-1}}\hat{w})e^{xz} + 2kz^{-2k}\hat{w}e^{xz}.\end{aligned}$$

Dividing from above equality the factor e^{xz}, the result of the proposition is obtained, and thus completes the proof. □

Further, we know from proposition 4 that the equivalent form of eq.(3.4), $\partial_{t^*_{1,-(l-1)}}\phi = 0$, implies the constraints on wave function, $\partial_{t^*_{1,-(l-1)}}\hat{w} = 0$, specifically. So it is very natural to express these constraints on the wave function by means of the τ function. By proceeding in this way, the explicit form of the Virasoro generators will be obtained as follows.

Lemma 2. *The action of additional symmetries on $\hat{w}$ can be expressed as a special form of*

$$(\partial_{t^*_{m,n}}\hat{w}) = f(t;z)\frac{\tilde{\tau}(t,z)}{\tau(t)}, \tag{4.3}$$

where $f(t;z) = g_1(t_{2n-1} \to t_{2n-1} - \frac{2}{(2n-1)z^{2n-1}}; \tilde{\tau}(t,z)) - g_1(t;\tau(t)) \equiv \tilde{g}(t;z) - g(t)$, *which is called a similarity shifted function.*

Proof By a straightforward calculation, we have

$$\begin{aligned}(\partial_{t^*_{m,n}}\hat{w}) &= \left(\partial_{t^*_{m,n}}\frac{\tilde{\tau}(t,z)}{\tau(t)}\right) = \frac{\tau(t)\partial_{t^*_{m,n}}\tilde{\tau}(t,z) - \tilde{\tau}(t,z)\partial_{t^*_{m,n}}\tau(t)}{\tau^2(t)} \\ &= \left(\frac{\partial_{t^*_{m,n}}\tilde{\tau}(t,z)}{\tilde{\tau}(t,z)} - \frac{\partial_{t^*_{m,n}}\tau(t)}{\tau(t)}\right)\frac{\tilde{\tau}(t,z)}{\tau(t)} = f(t;z)\frac{\tilde{\tau}(t,z)}{\tau(t)},\end{aligned}$$

as required. This is the end of the proof. □

This lemma reminds us that we should transform the $(\partial_{t^*_{1,-(l-1)}}\hat{w})$ given by eq.(4.2) to be the form of eq.(4.3) in order to find $(\partial_{t^*_{1,-(l-1)}}\tau(t))$, and then find Virasoro generators. However, in general, only this is not enough to guarantee us to get the correct Virasoro generators. Normally, the form of $(\partial_{t^*_{m,n}}\hat{w})$ is not unique, as pointed out by Dickey [11] for the KP hierarchy, which is also true for the BKP hierarchy, because there exists a freedom of gauge transformation with constant coefficients. One has to choose a suitable gauge such that a good form of $(\partial_{t^*_{m,n}}\hat{w})$ can be reached, and then the latter can lead to the correct Virasoro generators by means of $(\partial_{t^*_{m,n}}\tau)$. In particular, for the case of $n \geq 0$, it is very complicated to get a general simple expression of $(\partial_{t^*_{1,n+1}}\hat{w})$ for the KP hierarchy [11] and BKP hierarchy. However, it is simpler for the $n < 0$. This is a reason for us to only study $(\partial_{t^*_{1,-(l-1)}}\hat{w})$ in the last proposition 4 and to study $(\partial_{t^*_{1,-(l-1)}}\tau(t))$ in the sequent proposition with $l \geq 2$.

Lemma 3. *Suppose*

$$L_{-k} = \frac{1}{2}\sum_{n=k+1}^{\infty}(2n-1)t_{2n-1}\frac{\partial}{\partial t_{2n-2k-1}} + \frac{1}{8}\sum_{n+m=k+1}(2n-1)(2m-1)t_{2n-1}t_{2m-1}, \tag{4.4}$$

where n and m in the second term take value from 1 to k under the condition of $n+m = k+1$, then the Virasoro commutation relations

$$[L_{-k}, L_{-l}] = (-k+l)L_{-(k+l)} \tag{4.5}$$

hold for integers $k, l \geq 1$.

Proposition 5. *If L^l satisfies eq.(3.4), i.e., L is independent of $t^*_{1,-(l-1)}$, $l = 2k$, $k = 1, 2, 3, \cdots$, then the Virasoro constraints imposed on the τ function of the BKP hierarchy are*

$$L_{-k}\tau = 0, \tag{4.6}$$

where L_{-k} are Virasoro generators given by eq.(4.4).

Obviously, let $k = 1$, $L_{-1}\tau = 0$ is consistent with the corollary 1. The $\frac{1}{k}L_{-k}$ gives the same result as that obtained in Ref. [13] using a completely different approach. In particularly, the Virasoro generators for the BKP are indeed different with ones [19] of the KP hierarchy.

Proof For convenience, we mark the four terms of $(\partial_{t^*_{1,-(l-1)}}\hat{w})$ in eq.(4.2) by $a)$, $b)$, $c)$, $d)$, respectively. The proof has the following steps. In this proof, $\tilde{\tau} = \tilde{\tau}(t,z)$.

1). First of all, we try to construct the similarity shifted function structure in two terms, $a)$ and $c)$, because only they have the derivatives of τ and $\tilde{\tau}$. A direct calculation shows

$$a)\equiv -2z^{-2k+1}\frac{\partial \hat{w}}{\partial z}=-2z^{-2k+1}\frac{1}{\tau}\frac{\partial \tilde{\tau}}{\partial z}=-\frac{4}{\tau}\sum_{n=k+1}^{\infty}\frac{1}{z^{2n-1}}\frac{\partial \tilde{\tau}}{\partial t_{2n-2k-1}}, \tag{4.7}$$

and

$$\begin{aligned} c)\equiv & 2\sum_{n=k+1}(2n-1)t_{2n-1}(\frac{\partial}{\partial t_{2n-2k-1}}\frac{\tilde{\tau}}{\tau}) \\ &=\frac{2}{\tau}\sum_{n=k+1}^{\infty}(2n-1)t_{2n-1}\frac{\partial \tilde{\tau}}{\partial t_{2n-2k-1}}-\frac{2\tilde{\tau}}{\tau^2}\sum_{n=k+1}^{\infty}(2n-1)t_{2n-1}\frac{\partial \tau}{\partial t_{2n-2k-1}}. \end{aligned}$$

For the third term, we try to make a similarity shifted function deliberately by insertion of one term, and thus

$$\begin{aligned} c)=&(\frac{2}{\tilde{\tau}}\sum_{n=k+1}^{\infty}(2n-1)(t_{2n-1}-\frac{2}{(2n-1)z^{2n-1}})\frac{\partial \tilde{\tau}}{\partial t_{2n-2k-1}}-\frac{2}{\tau}\sum_{n=k+1}^{\infty}(2n-1)t_{2n-1}\frac{\partial \tau}{\partial t_{2n-2k-1}})\frac{\tilde{\tau}}{\tau} \\ &+\frac{4}{\tau}\sum_{n=k+1}^{\infty}\frac{1}{z^{2n-1}}\frac{\partial \tilde{\tau}}{\partial t_{2n-2k-1}}. \end{aligned} \tag{4.8}$$

Further,

$$\begin{aligned} a)+c)=&\Big(\frac{2}{\tilde{\tau}}\sum_{n=k+1}^{\infty}(2n-1)(t_{2n-1}-\frac{2}{(2n-1)z^{2n-1}})\frac{\partial \tilde{\tau}}{\partial t_{2n-2k-1}} \\ &-\frac{2}{\tau}\sum_{n=k+1}^{\infty}(2n-1)t_{2n-1}\frac{\partial \tau}{\partial t_{2n-2k-1}}\Big)\frac{\tilde{\tau}}{\tau} \end{aligned} \tag{4.9}$$

possess one similarity shifted function as we expected.
2)For the b)+d), it can not be transformed to a similarity shifted function if we only make a shift $t_{2n-1}\to t_{2n-1}-\frac{2}{(2n-1)z^{2n-1}}$ in b), similar to what like we have done in c). So we have to try it by using its product, as the second simplest case. To do this, by rewriting b) as a symmetrical form, and then using one identity, we get

$$\begin{aligned} b)\equiv & -2\sum_{n=1}^{k}z^{2n-2k-1}(2n-1)t_{2n-1}\hat{w} \\ =&-(\sum_{n+m=k+1}(2n-1)(2m-1)t_{2n-1}\frac{\hat{w}}{(2m-1)z^{2m-1}}+\sum_{n+m=k+1}(2n-1)(2m-1)t_{2m-1}\frac{\hat{w}}{(2n-1)z^{2n-1}}) \\ =&-\frac{1}{2}\Big(-\sum_{n+m=k+1}(2n-1)(2m-1)(t_{2m-1}-\frac{2}{(2m-1)z^{2m-1}})(t_{2n-1}-\frac{2}{(2n-1)z^{2n-1}}) \\ &+\sum_{n+m=k+1}(2m-1)(2n-1)t_{2n-1}t_{2m-1}\Big)\hat{w}-2\sum_{n+m=k+1}\frac{\hat{w}}{z^{2(n+m)-2}} \\ =&\frac{1}{2}\Big(\sum_{n+m=k+1}(2m-1)(2n-1)(t_{2m-1}-\frac{2}{(2m-1)z^{2m-1}})(t_{2n-1}-\frac{2}{(2n-1)z^{2n-1}}) \end{aligned}$$

$$-\sum_{n+m=k+1}(2m-1)(2n-1)t_{2n-1}t_{2m-1}\Big)\hat{w}-2kz^{-2k}\hat{w}. \tag{4.10}$$

Therefore,

$$\begin{aligned} b)+d) =&\frac{1}{2}\Big(\sum_{n+m=k+1}(2n-1)(2m-1)(t_{2m-1}-\frac{2}{(2m-1)z^{2m-1}})(t_{2n-1}-\frac{2}{(2n-1)z^{2n-1}})\\ &-\sum_{n+m=k+1}(2m-1)(2n-1)t_{2n-1}t_{2m-1}\Big)\frac{\tilde{\tau}}{\tau}. \end{aligned} \tag{4.11}$$

3) Taking a) + c) in eq.(4.9) and b) + d) in eq.(4.11) into $(\partial_{t^*_{1,-(l-1)}}\hat{w})$ in eq.(4.2), then

$$\begin{aligned} (\partial_{t^*_{1,-(l-1)}}\hat{w}) =&\Big(\frac{2}{\tilde{\tau}}\sum_{n=k+1}^{\infty}(2n-1)(t_{2n-1}-\frac{2}{(2n-1)z^{2n-1}})\frac{\partial\tilde{\tau}}{\partial t_{2n-2k-1}}\\ &+\frac{1}{2}\sum_{n+m=k+1}(2n-1)(2m-1)(t_{2n-1}-\frac{2}{(2n-1)z^{2n-1}})(t_{2m-1}-\frac{2}{(2m-1)z^{2m-1}})\Big)\frac{\tilde{\tau}}{\tau}\\ &-\Big(\frac{2}{\tau}\sum_{n=k+1}^{\infty}(2n-1)t_{2n-1}\frac{\partial\tau}{\partial t_{2n-2k-1}}+\frac{1}{2}\sum_{n+m=k+1}(2n-1)(2m-1)t_{2n-1}t_{2m-1}\Big)\frac{\tilde{\tau}}{\tau} \end{aligned} \tag{4.12}$$

On the one side, taking into account an equivalent form of the eq.(3.4), i.e., $\partial_{t^*_{1,-l-1}}\phi=0$, and the lemma 2, we have

$$\partial_{t^*_{1,-(l-1)}}\hat{w}=\Big(\frac{\partial_{t^*_{1,-l-1}}\tilde{\tau}}{\tilde{\tau}}-\frac{\partial_{t^*_{1,-l-1}}\tau}{\tau}\Big)\frac{\tilde{\tau}}{\tau}=0, \tag{4.13}$$

and then deduce

$$(\partial_{t^*_{1,-l-1}}\tau)=c_1\tau. \tag{4.14}$$

where c_1 is a constant. On the other side, comparing eq.(4.12) and eq.(4.13) infers

$$\begin{aligned} (\partial_{t^*_{1,-(l-1)}}\tau) =&\Big(2\sum_{n=k+1}^{\infty}(2n-1)t_{2n-1}\frac{\partial\tau}{\partial t_{2n-2k-1}}+\frac{1}{2}\sum_{n+m=k+1}(2n-1)(2m-1)t_{2n-1}t_{2m-1}\tau\Big)+c_2\tau\\ =&4L_{-k}\tau+c_2\tau \end{aligned} \tag{4.15}$$

with an arbitrary constant c_2, and L_{-k} as we expected. Therefore, the Virasoro constraints on the τ function

$$L_{-k}\tau=0$$

is obtained from eqs.(4.14) and (4.15) with $c_1=c_2$. This is the end of the proof. □

5. Conclusions and Discussions

We have studied the applications of the additional symmetry flows of the BKP hierarchy, and thus provided the following main results:

- 1) the action of the special additional symmetry flows on L^l and the string equation in proposition 3;
- 2) the explicit forms of the actions of the additional symmetry flows on the wave function $(\partial_{t^*_{1,-(l-1)}}\hat{w})$ in proposition 4;

- 3) the explicit forms of the negative Virasoro generators and the Virasoro constraints on the τ function of the BKP hierarchy in proposition 5.

In addition, the similarity shifted function $f(t;z)$ in $(\partial_{t^*_{m,n}}\hat{w})$ given by lemma 2 is also crucial to get the action of the additional symmetry flows on the τ function. Our route is *additional symmetry → additional symmetry flow equations eq.(3.4) associated with* $A_{1,-(l-1)}$ *→actions of the additional symmetry flows on the wave function → Virasoro constraints on the* τ *function → Virasoro generators.*

For the further research related to this topic, the extension of the additional symmetry and its associated structures for the multi-component BKP hierarchy [20] would be very interesting and relevant although very complicated. Moreover, it is also an interesting problem to calculate out the whole set of L_k for the Virasoro constraints and W_n for the W-constraints for the BKP hierarchy.

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References

[1] A. Yu. Orlov, E. I. Schulman, Additional symmetries of integrable equations and conformal algebra reprensentaion, Lett. Math. Phys. 12, 171-179(1986).
[2] M. Douglas,Strings in less than one dimension and the generalized KdV hierarchies Phys. Lett. B238, 176-180(1990).
[3] A. Morozov, Integrability and Matrix Models, Phys.Usp. 37, 1-55(1994). (also see arXiv:hep-th/9303139,149pages).
[4] P. Van Moerbke, Integrable foundation of string theory,in *Lectures on integrable systems,P.163-267*(World Scientific, Singapore, 1994)
[5] L. A. Dickey, Lectures on classical w-algebras, Acta Applicandae Mathematicae 47, 243-321(1997).
[6] E. Date, M. Kashiwara, M. Jimbo, T. Miwa, in *Nonlinear Integrable Systems- Classical and Quantum Theory*, edited by M. Jimbo and T. Miwa (World Scientific, Singapore, 1983) p. 39-119.
[7] L. A. Dickey, Soliton Equations and Hamiltonian Systems (2nd Edition)(World Scintific, Singapore, 2003).
[8] M. Adler, T. Shiota, P. van Moerbeke, A Lax representation for the Vertex operator and the central extension, Comm. Math. Phys. 171, 547-588(1995).
[9] M. Adler, T. Shiota, P. van Moerbeke, From the w_∞-algebra to its central extension: a τ-function approach, Phys. Lett. A 194, 33-43(1994).
[10] L. A. Dickey, On additional symmetries of the KP hierarchy and Sato's Backlund transformation, Comm. Math. Phys. 167, 227-233(1995).
[11] L. A. Dickey, Additional symmetries of KP, Grassmannian, and the string equation, Mod. Phys. Lett. A8, 1259-1272(1993).
[12] E. Date, M. Kashiwara, T. Miwa, Transformation groups for soliton equations. II.Vertex operators and τ functions, Proc. Japan Acad. Ser. A 57, 387-392(1981); E. Date, M. Jimbo, M. Kashiwara, T. Miwa, Transformation groups for soliton equations. IV. A new hierarchy of soliton equations of KP-type, Physica D4, 343-365(1981/82)
[13] Johan van de Leur, The n-th reduced BKP hierarchy, the string equation and $BW_{1+\infty}$-constraints, Acta Applicandae Mathematicae 44, 185-206(1996)(see also arXiv:hep-th/9411067).
[14] Johan van de Leur, The Adler-Shiota-van Moerbeke formula for the BKP hierarchy, J. Math. Phys. 36, 4940-4951(1995)
[15] K. Takasaki, Quasi-classical limit of BKP hierarchy and W-infnity symmetris, Lett. Math. Phys. 28, 177-185(1993).

[16] M. H. Tu, On the BKP hierarchy: Additional symmetries, Fay identity and Adler-Shiota- van Moerbeke formula, Lett. Math. Phys. 81, 91-105(2007)

[17] Jingsong He, Kelei Tian, Angela Foerster and Wen-xiu Ma, Additional Symmetries and String Equation of the CKP Hierarchy, Lett. Math. Phys. 81, 119-134 (2007).

[18] M. Adler, P. van Moerbeke, A matrix solution to two-dimensional W_p-gravity, Comm. Math. Phys. 147, 25-56(1992).

[19] M. Fukuma, H. Kawai1, and R. Nakayama, Infinite Dimensional Grassmannian Structure of Two-Dimensional Quantum Gravity, Commun. Math. Phys. 143, 371-403 (1992).

[20] K. Takasaki, Dispersionless Hirota equations of two-component BKP hierarchy, SIGMA 2, paper 057, 22pages(2006).